一. Basic Concepts

1. Triangle Inequality

$$|z_1+z_2| \le |z_1|+|z_2|$$

Proof: $|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$
$= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\bar{z}_1 z_2$
$= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\overline{z_1\bar{z}_2}$
$= |z_1|^2+|z_2|^2+2\,\mathrm{Re}(z_1\bar{z}_2)$
$\le |z_1|^2+|z_2|^2+2|z_1\bar{z}_2| = (|z_1|+|z_2|)^2$

2. Argument & Modulus

If $z = x+yi \longrightarrow \begin{cases} x & \text{Real part} \\ y & \text{Imaginary part} \end{cases}$

$\theta = \mathrm{Arg}(z) = \arctan\frac{y}{x}$ $\qquad |z| = \sqrt{x^2+y^2}$

$\theta + 2k\pi$ can also be a valid argument of a complex number

$$z = r\cos\theta + r\sin\theta\cdot i$$

$|z_1 z_2| = |z_1||z_2|$ $\quad \arg(z_1 z_2) = \arg(z_1)+\arg(z_2)$ easily proved by Euler's formula

3. Conjugate

$\bar{z} = x-yi$, $\arg\bar{z} = -\arg z$, $\bar{z}_1+\bar{z}_2 = \overline{z_1+z_2}$ $\quad \overline{z_1 z_2} = \bar{z}_1\bar{z}_2$

$\mathrm{Re}\,z = \frac{1}{2}(z+\bar{z})$, $\mathrm{Im}\,z = \frac{1}{2i}(z-\bar{z})$, $z\bar{z} = |z|^2$

4. Distance

$d(z_1,z_2) = |z_1-z_2|$ Euclideam distance.

So, $d(\cdot,\cdot)$ is defined on $\mathbb{C}$ and $d(\cdot,\cdot)$ is complete.

$d(\cdot,\cdot)$ is complete $\Leftrightarrow$ every Cauchy sequence converges

$\Leftrightarrow$ if $z_n \xrightarrow{n\to\infty} z$, then $\begin{cases} \mathrm{Re}\,z_n \xrightarrow{n\to\infty} \mathrm{Re}\,z \\ \mathrm{Im}\,z_n \xrightarrow{n\to\infty} \mathrm{Im}\,z \end{cases}$

5. Region

$[a,b]$ denotes the line segment connecting point a & b.

$$[a,b] = \left\{ a + \frac{\lambda - t_1}{t_2 - t_1}(b-a),\ t_1 \le \lambda \le t_2 \right\}.$$

↳ Polygon : $[a_1, a_2, \cdots, a_n] = \bigcup_{i=1}^{n-1} [a_i, a_{i+1}]$

6. Euler formula

$$e^{j\theta} = \cos\theta + j\sin\theta \Rightarrow re^{j\theta} = r(\cos\theta + j\sin\theta)$$

7. De Moivre formula

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

$$\hookrightarrow (e^{j\theta})^n = e^{jn\theta}$$

E.g. use $\cos\theta$ & $\sin\theta$ to represent $\cos 3\theta$ & $\sin 3\theta$

$$\cos 3\theta + j\sin 3\theta = (\cos\theta + j\sin\theta)^3$$

$$= \cos^3\theta + 3j\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - j\sin^3\theta$$

$$\therefore \cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

8. $D(a,r)$ is a disk centered at a, radius r.

$$= \{ z : |z-a| < r \}$$

9. A set is called <u>Connected</u> if it cannot be written as the union of seperate sets.

10. Polygonally Connected

a set $S \subseteq \mathbb{C}$ is said to be polygonally connected if each pair of points in S can be joined by a polygon that lies in S.

↓

11 Theorem:

If $\Omega \subseteq \mathbb{C}$, and Ω is open

Ω is connec $\Leftrightarrow$ Ω is polygonally connected.

Proof:

Ω is connected, open $a \in \Omega$. Let $\Omega_1 \subseteq \Omega$ and $\forall z \in \Omega_1$ can be connected from a with a polygon. Then, $\Omega_2 = \Omega / \Omega_1$.

If $z \in \Omega_1$, the $\exists$ open disk $D(z,r) \subseteq \Omega$. So, $\forall w \in D(z,r)$ can reach a by a polygon by $w \to z \to a$. So, $D(z,r) \subseteq \Omega_1$.

So, Ω_1 is open as Ω_1 is consisted by all open disks.

Similarily, we can prove with exactly same idea that Ω_2 is open.

Now, we have two disjoint open sets Ω_1 & Ω_2, and $\Omega = \Omega_1 \cup \Omega_2$.

To make sure Ω is connected, Ω_2 has to be $\emptyset$.

So: Connected open set $\Rightarrow$ Polygonally connected.

It's trivial to claim that polygonally connected is connected.

12. Jordan Curve.

Every continuous curve in $\mathbb{C}$ can be represented as

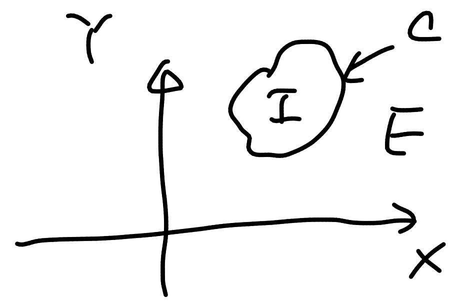
$$z(t) = x(t) + j\,y(t) \quad t \in [\alpha, \beta]$$

If $z(t_1) = z(t_2) \Rightarrow t_1 = t_2$, or say no intersection points,

then this curve is <u>Simple Curve</u>

△ If $z(\alpha) = z(\beta)$, this simple curve is <u>Jordan Curve</u> or Simple Closed Curve.

13 Jordan Curve Theorem.



A Jordan curve in $\mathbb{C}$ will separate the space to <u>Interior</u> <u>Exterior</u> and the curve itself.

① They all disjoint

② Interior is bounded

③ Exterior is unbounded

③ If a polygon P has two end points a, b

$a \in$ Interior, $b \in$ Exterior, then P must has intersections with Jordan Curve.